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Large-scale Structure and the Determination of H_0 from Gravitational Lens Time Delays

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Abstract

We analyse the effects of large-scale inhomogeneities upon the observables of a gravitational lens system, focusing on the issue of whether large-scale structure imperils the program to determine the Hubble parameter through measurements of time delays between multiple images in lens systems. We find that the lens equation in a spatially flat Robertson-Walker cosmology with scalar metric fluctuations is equivalent to that for the same lensing system in the absence of fluctuations, but with a different angular position of the source relative to the lens axis. Since the absolute position of the source is not observable, gravitational lens measurements cannot directly reveal the presence of large scale structure. Large-scale perturbations do not modify the functional relationship between observable lens parameters and the Hubble parameter, and therefore do not seriously affect the determination of H_0 from lens time delays.

Subject Headings: cosmology: large-scale structure of the universe — gravitational lensing

1 Introduction

Gravitational lenses are proving to be both intrinsically fascinating systems and valuable astrophysical tools that can help determine the fundamental cosmological parameters (for a review see Schneider et al. 1992). In principle, lens observations can provide estimates of the Hubble parameter $H_{\rm o}$ and the density parameter $\Omega_{\rm o}$ (Refsdal 1964). The consistency between gravitational lens estimates and more local determinations of these parameters would also constitute additional evidence in favor of the standard Friedmann-Robertson-Walker (FRW) cosmological models.

Measurements of time delays between multiple images in a gravitational lens system were first shown by Refsdal (1964) to provide a potentially direct determination of $H_{\rm o}$, independent of the standard cosmic distance ladder. If the structure of the deflector in a gravitational lens system is sufficiently well understood and its redshift known, then the relationship between the redshifts of the images and the deflector, and the images' relative angular positions, relative magnifications, and time delays, provide a determination of the deflector parameters, such as its mass, as well as of the Hubble parameter. To date, the double quasar 0957+561, the first lens system discovered, is the only one for which a time delay has been reliably measured (Vanderriest et al. 1989, Lehár et al. 1992, Press et al. 1992), and a value for $H_{\rm o}$ derived (Falco et al. 1991, Rhee 1991, Roberts et al. 1991). For this system, the estimated errors in $H_{\rm o}$ are around 10% to 30%, due to uncertainties in the density parameter $\Omega_{\rm o}$, in the relative angular positions of the images and the deflecting galaxy, and especially in the lens model parameters. It is reasonable to expect that measurements of time delays in other gravitational lens systems that have simpler structure than 0957+561 will improve this estimate in the near future, and provide a valuable tool for cosmology.

The aim of this article is to discuss whether the existence of large-scale structure in the Universe compromises the program to use time delay measurements to determine the Hubble parameter. In principle, inhomogeneities in the mass distribution along the photon paths affect the observables of a gravitational lens system, and if not properly taken into account as part of the model for the lens system, they could systematically bias the determination of $H_{\rm o}$ from lens time delays. Among others, Alcock & Anderson 1985, 1986, Watanabe, Sasaki, & Tomita 1992, Sasaki 1993, and Seljak 1994 have estimated the effect of departures from a FRW cosmology on the time delay between multiple images in a lens system. (Other recent studies of light propagation in perturbed FRW cosmologies include Durrer 1994 and Pyne and Birkinshaw 1994.)

There are two effects of large-scale structure on the relation between lens time delays and $H_{\rm o}$. First, long-range fluctuations in the gravitational potential near the line of sight to a lensed QSO can affect the distance measure, and thus the proportionality between the measured lens-induced time delay and $H_{\rm o}$. The time delay is proportional to $D_{\rm d}D_{\rm s}/D_{\rm ds} \sim H_{\rm o}^{-1}$, where $D_{\rm d}$ and $D_{\rm s}$ are the distances from the observer to the lens deflector and the source respectively, and $D_{\rm ds}$ is the distance between deflector and source. Here, the central issue of debate is what distance D is appropriate: the FRW angular-diameter distance, the Dyer-Roeder (Dyer and Roeder 1972) distance (for an empty or partially filled beam), or some other distance measure which also takes into account the effects of large-scale shear.

The second effect of large-scale perturbations is a *direct* contribution to the lens time delay, in addition to that arising from the lens itself. (This direct contribution may be present even if the lens-induced delay vanishes in the absence of perturbations; it is therefore conceptually distinct from the indirect distance-measure effect.)

This paper is primarily concerned with the latter, direct effect of perturbations on lens time delays. Our main conclusion is that, while large-scale inhomogeneities do have an effect upon time delays and other observables in a gravitational lens system, they do not compromise the program to extract from them the value of H_o. More precisely, the lens equation in the presence of scalar metric fluctuations is the same as that describing an identical lens system in the absence of fluctuations but with an (unobservably) different absolute angular position of the source. It is thus observationally impossible to distinguish time delays induced by large-scale structure from intrinsic delays due to the lens itself. The important corollary is that, to leading order in the fluctuations, the relationships between the observables of the lens system, the Hubble parameter, and the lens model parameters in the presence of scalar metric fluctuations are the same as if the inhomogeneities were absent, modulo distance measure effects. We also show that for large-scale, small-amplitude density perturbations, the modification of the distance measure from the FRW angular-diameter distance is small.

Taken together, these results imply that large-scale structure does not imperil the program to determine H_o from time delay measurements in gravitational lenses. On the other hand, it also implies that one cannot use time delay measurements to detect or constrain large-scale inhomogeneities in the Universe, once H_o is determined reliably by other means. It was suggested a few years ago (Allen 1989) that time delay measurements in gravitational lenses could serve as gravitational wave detectors. The same technique could in principle have been extended to probe large-scale inhomogeneities in the matter distribution (Frieman & Turner 1989, unpublished). We have argued, however, (Frieman, Harari, & Surpi 1994) that it is not observationally possible to distinguish the time delay induced by gravity waves (tensor metric fluctuations) from the intrinsic time delay originating in the lens geometry. Here we extend this result to scalar metric fluctuations of a FRW spacetime (arising from matter-density fluctuations), using the same technique based upon Fermat's principle in curved space-time. Our conclusions are similar to, but our methods differ from, those of Frieman, Kaiser, & Turner 1990 (unpublished). Our main result is that the lens equation in the presence of scalar metric fluctuations is effectively the same (to the relevant order of approximation) as in the absence of fluctuations, but with the lens system having a different alignment between observer, deflector, and source.

2 The Lens Equation and Large-scale Structure

Consider a thin, stationary gravitational lens, embedded in a spatially-flat FRW cosmology, with scalar metric perturbations representing large-scale matter-density fluctuations. We assume a weak gravitational field $U(\boldsymbol{x}, t)$ for the deflector in the lens system and small-amplitude metric inhomogeneities, and we work to first order in both the large-scale metric fluctuation amplitude $h_{\mu\nu}$ and the deflector potential U (which implies first order in α,

the deflection angle imprinted on the light rays). We use the longitudinal or conformal-Newtonian gauge, and let Greek indices μ, ν run from 0 to 3 while Latin indices i, j run from 1 to 3; we also set the speed of light $c = 1$. Defining the conformal time $d\eta = dt/a(t)$, with t the proper time and $a(t)$ the cosmic scale-factor, the metric is

$$ds^2 = a^2(\eta)[(1 + 2(U + \phi))d\eta^2 - (1 - 2(U + \phi))\delta_{ij}dx^i dx^j] \ , \tag{1}$$

where $\phi(\boldsymbol{x}, \eta)$ completely describes the scalar metric fluctuation for a matter-dominated universe, and is equal to a gauge-invariant variable; it satisfies the relativistic generalization of the Poisson equation. For $\Omega_{\rm o} = 1$, the conformal time and scale-factor can be normalized so that $t = 2\eta^3/3H_{\rm o}$ and $a(\eta) = 2\eta^2/H_{\rm o}$, where $\eta_{\rm o} = 1$ and subscript zero indicates the present epoch. For the growing mode of adiabatic scalar density fluctuations in a matter-dominated ($\Omega_{\rm o} = 1$) universe, the metric perturbation amplitude is time-independent, $\phi(\eta, \boldsymbol{x}) = \phi_{\rm o}(\boldsymbol{x})$ (e.g., Mukhanov et al. 1992). The results below can easily be extended to the non-flat case $\Omega_{\rm o} \neq 1$.

Without loss of generality, we can place the observer at the origin of coordinates and the z-axis coincident with the lens axis (the line that joins observer and deflector); the deflector is located at $\boldsymbol{D} = (0, 0, r_{\rm d})$ and the source at $\boldsymbol{S} = (x_{\rm s}, 0, r_{\rm s})$, see Figure 1. To avoid confusion with redshift, we use r to denote the value of the coordinate z along the lens axis. We denote angular positions by two-component vectors, defined by the (x, y) components along the photon paths in the plane perpendicular to the lens axis. The indices a, b, running from 1 to 2, denote these components. Thus we characterize the absolute angular position of the source relative to the lens axis by the vector $\boldsymbol{\beta} = (x_{\rm s}/r_{\rm s}, 0)$ (these are the components at the source position, $z = r_{\rm s}$).

From Eq. (1), if the spatial photon trajectories are known, one can evaluate the conformal time of travel by integration along r (the coordinate along the lens axis),

$$\eta \approx \int_0^{r_{\rm s}} dr \left[1 + \frac{1}{2}\left(\frac{dx}{dr}\right)^2 + \frac{1}{2}\left(\frac{dy}{dr}\right)^2 - 2(U + \phi)\right] \ . \tag{2}$$

The first three terms in the integrand of Eq. (2) are the geometric contributions to the travel time, while the last contains the gravitational potential contributions from the lens deflector and large-scale metric perturbations.

We now determine the photon paths by implementing Fermat's principle (Blandford & Narayan 1986, Kovner 1990, Nityananda and Samuel 1992). We first determine the time of travel along null trial paths of the metric (1). Each trial path is composed of two segments, one from the source to the deflector plane, and another from the deflector plane to the observer, as appropriate for a thin lens. Each segment is a solution of the geodesic equations for the metric (1) *neglecting* the gravitational potential U of the deflector (which is taken into account through the bending at the deflector), but *including* the effect of the large-scale metric fluctuation ϕ. We further require that the paths built in this way are null paths of the full metric (1) (i.e., they satisfy $ds^2 = 0$ and Eq. (2)) but not necessarily geodesics. Along them, we evaluate the conformal travel time from source to observer. Fermat's principle states that null geodesics are those null paths for which the arrival time is an extremum. Making the conformal travel time an extremum leads to the lens equation, from which the

relationships among the apparent angular positions of the multiple images, their time delays, relative magnifications, and the lens parameters can be read off.

The affine parameter for the photon geodesics of the metric (1), neglecting the deflector gravitational potential U, may be written as $\lambda = r + \mathcal{O}(\phi)$. To leading order in ϕ, the corresponding geodesic equations may then be integrated to obtain

$$\frac{d\boldsymbol{x}}{dr} = \boldsymbol{\epsilon} + \boldsymbol{\Delta}(r) \quad , \tag{3}$$

(recall that $\boldsymbol{x}(r)$ denotes the vector whose (x, y) components give the photon trajectory parametrized in terms of the distance along the lens axis) where $\boldsymbol{\epsilon}$ denotes an arbitrary integration constant. For the scalar growing mode, $\boldsymbol{\Delta}$ is given by

$$\Delta^a(r) = -2 \int^r dr \phi_{,a} \quad , \tag{4}$$

where a comma denotes an ordinary derivative in the FRW metric. Here, we have assumed that these trajectories form small angles with the lens axis, and we work to first order in ϕ and in the trajectory angle from the lens axis. That is, we can expand the function in eqn.(4) around the lens axis, $\Delta^a(x^b, r) = \Delta^a(r) + \Delta^a_{,b} x^b(r) + ...$, where $\Delta^a(r) \equiv \Delta^a(0, r)$ is the function evaluated along the lens axis. Since $\Delta^a(r) \propto \phi_{,a}$, the approximation in eqns.(3) and (4), i.e., dropping terms of order $\Delta^a_{,b}$, corresponds to keeping only first derivatives (gradients) of the potential, neglecting terms of order $\partial^2\phi/\partial x^a \partial x^b$. This means that we are not including the relative focusing due to large-scale fluctuations. For typical gravitational lenses, the maximum transverse separation between the image paths is $\xi \sim D\theta \sim 10 - 20$ kpc, much smaller than the wavelengths $\lambda \gtrsim 10$ Mpc of the large-scale perturbations we are interested in. Thus, the second derivative terms we are neglecting are suppressed compared to the first derivative terms by a factor $\xi/\lambda \lesssim 10^{-3}$. We shall use this approximation consistently throughout. As a result, the integrand in Eq. (4) is to be evaluated along the lens axis, which one can think of as a fiducial or 'mean' photon path.

We now use Eq.(3), with appropriate values for the integration constants $\boldsymbol{\epsilon}$ in each segment $0 < r < r_{\rm d}$ and $r_{\rm d} < r < r_{\rm s}$, to build a zig-zag trajectory that starts at $\boldsymbol{S}$, is deflected at $r = r_{\rm d}$ (the deflector plane), and arrives at the observer at the origin. There is a family of trajectories satisfying these focusing conditions, which we choose to parametrize in terms of the apparent angular position $\boldsymbol{\theta}$ of the source image relative to the deflector (since this is an observable quantity). To evaluate $\boldsymbol{\theta}$, we use again Eq. (3) with appropriate boundary conditions to evaluate the photon trajectory $\boldsymbol{x}_{\rm d}(r)$ that arrives directly from the deflector to the observer simultaneously with the source image. The apparent angular position of the image relative to the deflector as seen by the observer is then given by

$$\boldsymbol{\theta} \equiv \left.\frac{d\boldsymbol{x}}{dr}\right|_{r=0} - \left.\frac{d\boldsymbol{x}_{\rm d}}{dr}\right|_{r=0} \quad . \tag{5}$$

The family of trajectories that meet the focusing conditions at the source and the observer, parametrized in terms of $\boldsymbol{\theta}$, satisfies

$$\begin{aligned} \frac{d\boldsymbol{x}}{dr} &= \boldsymbol{\theta} - \int_0^{r_{\rm d}} \frac{\boldsymbol{\Delta}(r)}{r_{\rm d}}\, dr + \boldsymbol{\Delta}(r) \qquad \text{if} \quad r < r_{\rm d} \\ \frac{d\boldsymbol{x}}{dr} &= -\frac{r_{\rm d}}{r_{\rm ds}}\, \boldsymbol{\theta} + \frac{r_{\rm s}}{r_{\rm ds}}\, \boldsymbol{\beta} - \int_{r_{\rm d}}^{r_{\rm s}} \frac{\boldsymbol{\Delta}(r)}{r_{\rm ds}}\, dr + \boldsymbol{\Delta}(r) \qquad \text{if} \quad r > r_{\rm d} \end{aligned} \tag{6}$$

where $r_{\rm ds} \equiv r_{\rm s} - r_{\rm d}$. The deflection angle imprinted by the lens upon the trajectory at the deflector plane ($r = r_{\rm d}$), which we denote by $\boldsymbol{\alpha}$, is given by

$$\boldsymbol{\alpha} \equiv \left.\frac{d\boldsymbol{x}}{dr}\right|_{r=r_{\rm d}^-} - \left.\frac{d\boldsymbol{x}}{dr}\right|_{r=r_{\rm d}^+} \quad . \tag{7}$$

Using Eq. (6), we find

$$\boldsymbol{\alpha} = \frac{r_{\rm s}}{r_{\rm ds}}(\boldsymbol{\theta} - \boldsymbol{\beta}_{\rm eff}) \quad , \tag{8}$$

where the effective misalignment angle $\boldsymbol{\beta}_{\rm eff}$ has been defined as

$$\boldsymbol{\beta}_{\rm eff} \equiv \boldsymbol{\beta} + \boldsymbol{\beta}_{\rm LSS} \quad , \tag{9}$$

with

$$\boldsymbol{\beta}_{\rm LSS} = \frac{1}{r_{\rm d}} \int_0^{r_{\rm d}} \boldsymbol{\Delta}(r) dr - \frac{1}{r_{\rm s}} \int_0^{r_{\rm s}} \boldsymbol{\Delta}(r) dr \quad . \tag{10}$$

Written this way, Eqn. (8) is identical in form to the equation relating the image angular position $\boldsymbol{\theta}$ with the source misalignment angle $\boldsymbol{\beta}$ and the deflection $\boldsymbol{\alpha}$ in the absence of density fluctuations, but now in terms of an effective misalignment $\boldsymbol{\beta}_{\rm eff}$. The geometric meaning of $\boldsymbol{\beta}_{\rm eff}$ is apparent from Eqn. (8): it is the source-deflector misalignment angle that an observer would infer from lens observations, assuming a homogeneous FRW spacetime, i.e., with no knowledge of large-scale perturbations. Alternatively, it is the source angular position that the observer would measure in the perturbed FRW universe in the limit that the lens mass vanishes ($\alpha \to 0$).

Now we evaluate the conformal time of travel by integrating Eq. (2) along the null trajectories of Eq. (6). It is useful to distinguish two contributions to the travel time, the geometric, $\eta_{\rm g}$, and the potential, $\eta_{\rm p}$. The former is easily evaluated:

$$\eta_{\rm g} = \int_0^{r_{\rm s}} dr \left[1 + \frac{1}{2}\left(\frac{dx}{dr}\right)^2 + \frac{1}{2}\left(\frac{dy}{dr}\right)^2\right] = r_{\rm s} + \frac{1}{2}\frac{r_{\rm d} r_{\rm s}}{r_{\rm ds}} \boldsymbol{\theta} \cdot (\boldsymbol{\theta} - 2\boldsymbol{\beta}) \quad . \tag{11}$$

Here, we have discarded a $\boldsymbol{\theta}$-independent term, irrelevant to extremizing the travel time. Note that Eq. (11) has the same form as the geometric contribution to the time of travel in the absence of density fluctuations. The fluctuations are taken into account, however, through the fact that $\boldsymbol{\theta}$ is the apparent angular position of the source image relative to the deflector in the presence of the metric perturbation ϕ. This is a crucial step that allows us to establish the equivalence between a lens in the presence of density fluctuations and a lens in the absence of fluctuations but with a different source alignment.

The evaluation of the gravitational potential contribution to the time of travel requires more care, so we divide it into two parts, the first due to the large-scale metric fluctuations, $\eta_{\rm p}^{\rm LSS}$, and the second due to the potential U of the deflector, $\eta_{\rm p}^{\rm GL}$. For the large-scale structure contribution, to first order in ϕ and θ we have

$$\eta_{\rm p}^{\rm LSS} = -2 \int_0^{r_s} dr \phi(x^a, r) \quad , \tag{12}$$

and the integration is taken along the unperturbed path, i.e., the trajectories of Eq. (6) with $\Delta^a = 0$. In the integrand, we can expand the metric amplitude as

$$\phi(x^a, r) = \phi(0, r) + \phi_{,a} x^a(r) + \phi_{,a,b} x^a(r) x^b(r) + \ldots \quad , \tag{13}$$

where $x^a(r)$ is given by integrating (6). In keeping with the approximation used in eqns.(3 and 4), we neglect terms beyond the first derivative on the RHS of eqn.(13). Substituting (13) into (12), integrating by parts, and retaining only θ^a-dependent terms, we find

$$\eta_{\rm p}^{\rm LSS} = -\boldsymbol{\theta} \cdot \left(\int_0^{r_{\rm d}} dr\ \boldsymbol{\Delta} - \int_{r_{\rm d}}^{r_{\rm s}} dr\ \frac{r_{\rm d}}{r_{\rm s}} \boldsymbol{\Delta} \right) = -\frac{r_{\rm d} r_{\rm s}}{r_{\rm ds}}\ \boldsymbol{\theta} \cdot \boldsymbol{\beta}_{\rm LSS} \quad , \tag{14}$$

with $\boldsymbol{\beta}_{\rm LSS}$ as defined in Eq. (10). As above, the middle terms in Eq.(14) are to be integrated along the fiducial mean path.

The last contribution to the time of travel, due to the gravitational potential U of the deflector, is a local effect in a thin lens. It results in a function $\psi(\boldsymbol{\xi})$, where $\boldsymbol{\xi}$ is the impact parameter. Integrating eqn.(6), one finds $\boldsymbol{\xi} = \boldsymbol{x}(r_{\rm d}) = r_{\rm d} \boldsymbol{\theta}$. Thus, η can be expressed in terms of the lens mass density projected on the lens plane, $\Sigma(\boldsymbol{\xi})$ (Schneider, Ehlers & Falco 1992),

$$\eta_{\rm p}^{\rm GL} = -2 \int_0^{r_{\rm s}} U\ dr = -4G \int d^2\xi'\ \Sigma(\boldsymbol{\xi'}) \ln \left| \frac{\boldsymbol{\xi} - \boldsymbol{\xi'}}{r_{\rm d}} \right| \equiv -\psi(\boldsymbol{\xi}) \quad . \tag{15}$$

Putting together the results from Eqs. (11), (14) and (15), the total conformal travel time for a source image, $\eta = \eta_{\rm g} + \eta_{\rm p}^{\rm LSS} + \eta_{\rm p}^{\rm GL}$, becomes (up to $\boldsymbol{\theta}$-independent terms)

$$\eta = r_{\rm s} + \frac{r_{\rm d} r_{\rm s}}{2 r_{\rm ds}}\ (\boldsymbol{\theta} - 2\boldsymbol{\beta}_{\rm eff}) \cdot \boldsymbol{\theta} - \psi(r_{\rm d} \boldsymbol{\theta}) \quad , \tag{16}$$

with $\boldsymbol{\beta}_{\rm eff}$ as defined in Eq. (10). The main conclusion of this article derives from equation (16), which may be thought of as an "equivalence principle" for gravitational lenses. The travel time (16) is equivalent to the time of travel in a lens system identical to the one considered from the outset, but with no metric perturbations and with a different source alignment given by $\boldsymbol{\beta}_{\rm eff}$ instead of $\boldsymbol{\beta}$. However, as noted above, $\boldsymbol{\beta}_{\rm eff}$ is precisely the mis-alignment angle the observer would infer from lens observations *without* taking into account the large-scale perturbations. Thus, the observer can self-consistently ignore the presence of metric perturbations from the outset and obtain the correct time delay. The second term in equation (16), which partially originates in the metric fluctuation ϕ, masquerades as an intrinsic geometric effect of the lens. The lens equations, deduced from the condition that η be an extremum under variations of θ for fixed source position, are the same in the two cases, so an observer measuring redshifts, relative angular positions, relative magnifications, and time delays, is unable to tell the two situations apart–we will show this explicitly below.

To apply the results above, we must translate the conformal time delay between two images at angular positions $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_2}$,

$$\Delta\eta = \eta(\boldsymbol{\theta_1}) - \eta(\boldsymbol{\theta_2}) = \frac{r_{\rm d} r_{\rm s}}{2 r_{\rm ds}} (\boldsymbol{\theta_1} - \boldsymbol{\theta_2}) \cdot (\boldsymbol{\theta_1} + \boldsymbol{\theta_2} - 2\boldsymbol{\beta}_{\rm eff}) - (\psi(\boldsymbol{\xi_1}) - \psi(\boldsymbol{\xi_2})) \quad , \tag{17}$$

into the proper time delay Δt measured by the observer. We treat the first and second terms on the r.h.s. of Eq. (17) separately. The second term, due to the deflector gravitational

potential, is a purely local effect, which occurs when the photon paths are close to the deflector. Thus, the passage from conformal time-delay to proper time-delay at the observer's position is simply given by the factor $(1+z_{\rm d})$, with $z_{\rm d}$ the deflector redshift, which accounts for the time dilation caused by the expansion of the Universe as the photons travel from the deflector to the observer. The first term in Eq. (17), on the other hand, is translated to the observer's proper time by multiplying it by the present scale factor $a_{\rm o}$ (a good approximation since $\Delta t << H_{\rm o}^{-1}$). In order to make contact with observations, it is best to express the result in terms of the source and deflector redshifts $z_{\rm s}, z_{\rm d}$. For the leading order contribution to the time delay, we evaluate the relationship between comoving coordinates and redshifts in the unperturbed FRW background (see discussion at the end of this section). For a photon emitted at coordinate distance $r_{\rm e}$ at time $t = t_{\rm e}$ and observed at $r = 0$ at time $t = t_{\rm o}$ in a matter-dominated spatially-flat FRW cosmology (e.g., Schneider et al. 1992),

$$r_{\rm e} = \frac{2}{H_{\rm o} a_{\rm o}} \frac{\sqrt{1+z} - 1}{\sqrt{1+z}} = \frac{D_{\rm e}}{a_{\rm e}} \quad , \tag{18}$$

where the last term defines the angular-diameter distance $D_{\rm e}$ to the event with comoving coordinate r_e. In terms of angular-diameter distances, the observer's proper time delay between the two images becomes

$$\Delta t = (1+z_{\rm d}) \left[\frac{D_{\rm d} D_{\rm s}}{D_{\rm ds}} (\boldsymbol{\theta_1} - \boldsymbol{\theta_2}) \cdot (\frac{\boldsymbol{\theta_1} + \boldsymbol{\theta_2}}{2} - \boldsymbol{\beta}_{\rm eff}) - (\psi(D_{\rm d}\boldsymbol{\theta}_1) - \psi(D_{\rm d}\boldsymbol{\theta}_2)) \right] \quad . \tag{19}$$

The appearance of the angular diameter distances $D_{\rm d}$, $D_{\rm s}$ and $D_{\rm ds}$ in this expression does not imply the need to independently determine the distance scale to the source and deflector: the combination $(1+z_{\rm d})D_{\rm d}D_{\rm s}/D_{\rm ds}$ should simply be taken as shorthand for

$$(1+z_{\rm d}) \frac{D_{\rm d} D_{\rm s}}{D_{\rm ds}} = \frac{2}{H_{\rm o}} \frac{(1-\sqrt{1+z_{\rm s}})(1-\sqrt{1+z_{\rm d}})}{\sqrt{1+z_{\rm d}} - \sqrt{1+z_{\rm s}}} \quad . \tag{20}$$

The lens equation, obtained from the requirement that the time of travel for each image be an extremum with respect to variations in θ (Fermat's principle) is now

$$\frac{\partial \psi}{\partial \boldsymbol{\theta}} = \frac{D_{\rm d} D_{\rm s}}{D_{\rm ds}} \, (\boldsymbol{\theta} - \boldsymbol{\beta}_{\rm eff}) \tag{21}$$

This agrees with Eq. (8), since the deflection angle imprinted by the lens is given as a function of the lens mass distribution by $\boldsymbol{\alpha} = \partial\psi/\partial\boldsymbol{\xi}$.

In the limit that the relative change induced by large-scale structure upon the angular separation between multiple images is small ($|\boldsymbol{\beta}_{\rm LSS}| << |\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2|$), use of the lens equation (21) allows the time delay to be written in a simpler form,

$$\Delta t = \Delta t^{\rm intrinsic} + \Delta t^{\rm LSS} \quad , \tag{22}$$

where $\Delta t^{\rm intrinsic}$ is the time delay due to the lens geometry evaluated in the absence of metric perturbations, and $\Delta t^{\rm LSS}$ is the lowest-order effect of the metric perturbations,

$$\Delta t^{\rm LSS} = -(1+z_{\rm d}) \frac{D_{\rm d} D_{\rm s}}{D_{\rm ds}} \, \boldsymbol{\beta}_{\rm LSS} \cdot (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2) \quad . \tag{23}$$

Here it suffices to evaluate the angular separation between the images in the absence of metric perturbations.

In the general case, we can use the lens equation (21) to rewrite the proper time delay for the image observed at angular position $\boldsymbol{\theta}_i$ in terms of lens observables as

$$\frac{\Delta t(\boldsymbol{\theta}_i)}{(1+z_d)D_d} = -\frac{\theta_i^2 D_{\rm s}}{2D_{\rm ds}} + \frac{\partial \chi}{\partial \boldsymbol{\theta}_i} \cdot \boldsymbol{\theta}_i - \chi(\boldsymbol{\theta}_i) \quad , \tag{24}$$

where $\chi(\boldsymbol{\theta}) = \psi(\boldsymbol{\theta})/D_d$ is the dimensionless lens potential. The fact that this expression does not contain β_{LSS}, and is therefore independent of ϕ, embodies our main conclusion: while it is clear, for instance from Eqs. (22) and (23), that large-scale inhomogeneities do influence the time delay between multiple images in a gravitational lens, at the same time they affect the other lens observables in such a way that they leave no *observable* tracks of their presence. That is, in Eq. (24), lens observations in principle provide the required distance factors and the angle $\boldsymbol{\theta}_i$, while modelling of the lens (based on an inferred velocity dispersion and the observed surface brightness distribution) provides a parametrized determination of χ, with no reference to large-scale structure. Thus, (24) is just the time delay that an observer with no knowledge of large-scale structure would use to constrain his or her lens model.

The final thread in the argument that large-scale fluctuations do not perturb the relation between $H_{\rm o}$ and measured lens time delays involves the distance measure. In eq.(18) and following, we used the angular-diameter distance for the FRW background. Since the matter in the universe along the line of sight to a lensed system is clumped, it is not clear that this is the appropriate distance measure to use, and there is a large literature that treats this thorny issue. Here, we note that by focusing on the effects of small-amplitude, large-scale perturbations, we may essentially sidestep this debate. For linear perturbations, one can expand the distance measure around the FRW background (e.g., Sasaki 1993). The distance measure perturbation is then proportional to the density perturbation averaged over the beam. It is clear that this average is generally less than of order the typical perturbation amplitude. Thus, for linear density perturbations, $\delta\rho/\rho \ll 1$, the distance measure perturbation is small, $\delta D/D \lesssim \delta\rho/\rho \ll 1$.

3 Conclusion

The measurement of time delays between multiple images in a gravitational lens system can yield an estimate of the Hubble constant, as well as the lens mass or other lens parameters. Indeed, once a model for the deflecting object is assumed, knowledge of the redshifts of the images and the deflector, of their apparent relative angular positions, relative magnifications, and time delay allow a determination of the Hubble constant, since $\Delta t \propto D_{\rm d}D_{\rm s}/D_{\rm ds} \propto H_{\rm o}^{-1}$.

For the lens equation in the presence of large-scale matter-density fluctuations, the perturbations appear only through the quantity $\boldsymbol{\beta}_{\rm LSS}$ of Eq. (10), which is simply added to the intrinsic lens misalignment $\boldsymbol{\beta}$, to give the effective misalignment angle $\boldsymbol{\beta}_{\rm eff} = \boldsymbol{\beta} + \boldsymbol{\beta}_{\rm LSS}$. Thus, matter-density fluctuations have exactly the same observational consequences as a change

in the (unobservable) misalignment angle between the source and the lens axis. The standard method to determine the Hubble constant H_o proceeds from the lens observables in the usual way (Schneider et al. 1992), as if the large-scale inhomogeneities were absent. Indeed, the unobservable effective misalignment $\boldsymbol{\beta}_{\rm eff}$ cancels out from the expressions that relate the angular separation between the images, $\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2$, and the time delay Δt (Cf. eqn.(24)).

Thus, large-scale matter-density fluctuations along the line of sight do not compromise the program to determine the value of the Hubble constant through time-delay measurements between multiple images in gravitational lenses. This also implies that lens time delay measurements are not likely to provide information about large-scale inhomogeneities in the matter distribution.

It is important to emphasize that this conclusion is limited to the effects of small-amplitude (i.e., linear), large-scale density fluctuations. The assumption of linearity enters in several places. First, it implied that the scalar potential ϕ is time-independent in a spatially flat FRW background. While this simplifies the calculation, it is not an essential ingredient. More important, it is only for small-amplitude density perturbations that one can justify using the angular-diameter distance rather than a perturbed distance measure. This does not preclude the possibility that non-linear structure on small scales (e.g. galaxies) could significantly perturb the distance measure. The assumption of 'large scale' entered when we neglected second derivatives of the potential: this restricts the treatment to perturbation wavelengths λ much larger than the maximum transverse path separation, $\xi \sim 10 - 20$ kpc. Observations of galaxy clustering indicate that fluctuations in the galaxy density are non-linear on scales less than several Mpc, so the restriction to linearity already implies that the assumption $\lambda \gg \xi$ is satisfied. Finally, we note that our conclusions are valid not only for scalar metric perturbations, but also for tensor metric fluctuations (gravitational waves), as shown in Frieman, Harari & Surpi 1994.

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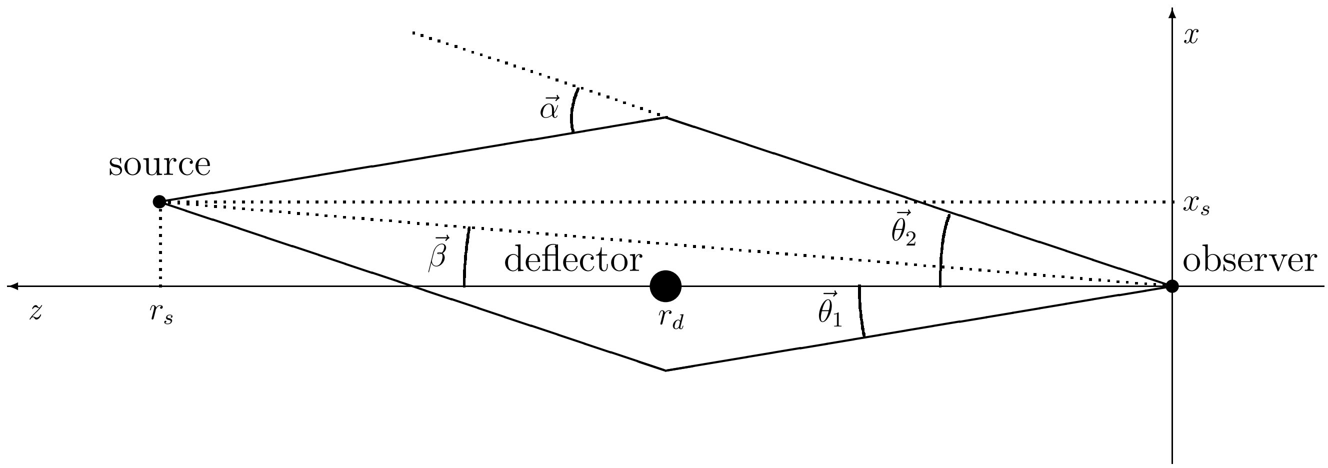

Figure 1: Gravitational lens geometry. The observer is at the origin of coordinates. The z-axis coincides with the lens axis, and the z-coordinate is labeled by r to avoid confusion with redshift. The source forms an angle $\vec{\beta}$ with respect to the lens axis. $\vec{\theta}_1$ and $\vec{\theta}_2$ are the apparent angular positions of the images and $\vec{\alpha}$ is the deflection angle.